Black holes, compact objects and solar system tests in nonrelativistic general covariant theory of gravity

Jared Greenwald,* V. H. Satheeshkumar,† and Anzhong Wang‡

GCAP-CASPER, Physics Department, Baylor University, Waco, TX 76798-7316, USA

(Dated: November 30, 2010)

We study spherically symmetric static spacetimes generally filled with an anisotropic fluid in the nonrelativistic general covariant theory of gravity. In particular, we find that the vacuum solutions are not unique, and can be expressed in terms of the $U(1)$ gauge field A. When solar system tests are considered, severe constraints on A are obtained, which seemingly pick up the Schwarzschild solution uniquely. In contrast to other versions of the Horava-Lifshitz theory, non-singular static stars made of a perfect fluid without heat flow can be constructed, due to the coupling of the fluid with the gauge field. These include the solutions with a constant pressure. We also study the general junction conditions across the surface of a star. In general, the conditions allow the existence of a thin matter shell on the surface. When applying these conditions to the perfect fluid solutions with the vacuum ones as describing their external spacetimes, we find explicitly the matching conditions in terms of the parameters appearing in the solutions. Such matching is possible even without the presence of a thin matter shell.

PACS numbers: 04.60.-m; 98.80.Cq; 98.80.-k; 98.80.Bp

I. INTRODUCTION

Recently, Horava proposed a theory of quantum gravity [1], motivated by the Lifshitz theory in solid state physics [2]. Due to several remarkable features, the Horava-Lifshitz (HL) theory has attracted a great deal of attention (see for example, [3] and references therein).

In the HL theory, Lorentz symmetry is broken in the ultraviolet (UV). The breaking manifests in the strong anisotropic scalings between space and time,

$$\mathbf{x} \to \ell \mathbf{x}, \quad t \to \ell^z t. \tag{1.1}$$

In $(3+1)$-dimensional spacetimes, the theory is power-counting renormalizable, provided that $z \geq 3$. At low energies, the theory is expected to flow to $z = 1$, whereby the Lorentz invariance is "accidentally restored." Such an anisotropy between time and space can be easily realized, when writing the metric in the Arnowitt-Deser-Misner (ADM) form [4],

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \quad (i,\ j = 1,2,3). \tag{1.2}$$

Under the rescaling (1.1) with $z = 3$ (a condition we shall assume in this paper), N, N^i and g_{ij} scale as,

$$N \to N,\ N^i \to \ell^{-2} N^i,\ g_{ij} \to g_{ij}. \tag{1.3}$$

The gauge symmetry of the system are the foliation-preserving diffeomorphisms Diff$(M,\ \mathcal{F})$,

$$\tilde{t} = t - f(t), \quad \tilde{x}^i = x^i - \zeta^i(t, \mathbf{x}), \tag{1.4}$$

for which N, N^i and g_{ij} transform as

$$\begin{aligned} \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\ \delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \end{aligned} \tag{1.5}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to the 3-metric g_{ij}, and $N_i = g_{ik} N^k$, etc. From these expressions one can see that the lapse function N and the shift vector N_i play the role of gauge fields of the Diff$(M,\ \mathcal{F})$ symmetry. Therefore, it is natural to assume that N and N_i inherit the same dependence on spacetime as the corresponding generators,

$$N = N(t), \quad N_i = N_i(t, x), \tag{1.6}$$

which is clearly preserved by the Diff$(M,\ \mathcal{F})$, and usually referred to as the projectability condition.

Due to the restricted diffeomorphisms (1.4), one more degree of freedom appears in the gravitational sector - the spin-0 graviton. This is potentially dangerous, and needs to decouple in the infrared (IR), in order to be consistent with observations. Whether it is the case or not is still an open question [5]. In particular, the spin-0 mode is not stable in the Minkowski background, in the original version of the HL theory [1] and in the Sotiriou, Visser and Weinfurtner (SVW) generalization [6, 7]. Although in the SVW setup it is stable in the de Sitter background [3]. In addition, non-perturbative analysis showed that it indeed decouples in the vacuum spherical static [5] and cosmological [8] spacetimes.

To overcome the problem, various models have been proposed [9]. In particular, Horava and Melby-Thompson (HMT) [10] recently proposed a version in which the spin-0 graviton is completely eliminated by introducing a Newtonian pre-potential φ and a local U(1) gauge field A, so that the foliation-preserving-

*Electronic address: Jared˙Greenwald@baylor.edu
†Electronic address: VH˙Satheeshkumar@baylor.edu
‡Electronic address: Anzhong˙Wang@baylor.edu

diffeomorphisms, Diff(M, $\mathcal{F}$), are extended to

$$U(1) \ltimes \text{Diff}(M,\ \mathcal{F}). \tag{1.7}$$

Effectively, the spatial diffeomorphism symmetries of general relativity are kept intact, but the time reparametrization symmetry is contracted to a local gauge symmetry [11]. The restoration of general covariance, characterized by Eq.(1.7), nicely maintains the special status of time, so that the anisotropic scaling (1.1) with $z > 1$ can still be realized. Under the Diff(M, $\mathcal{F}$), the fields A and φ transform as,

$$\begin{aligned} \delta A &= \zeta^i \partial_i A + \dot{f} A + f \dot{A}, \\ \delta\varphi &= f\dot{\varphi} + \zeta^i \partial_i \varphi, \end{aligned} \tag{1.8}$$

while under the local $U(1)$, they, together with g_{ij}, transform as

$$\begin{aligned} \delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha, \\ \delta_\alpha N_i &= N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0 = \delta_\alpha N, \end{aligned} \tag{1.9}$$

where α is the generator of the local $U(1)$ gauge symmetry. A remarkable by-production of this "non-relativistic general covariant" setup is that it forces the coupling constant λ, introduced originally to characterize the deviation of the kinetic part of the action from GR [1], to take exactly its relativistic value $\lambda = 1$. A different view can be found in [12].

In this paper, we investigate systematically the spherically symmetric spacetimes in the HMT setup. In particular, after briefly reviewing the theory in Sec. II, we develop the general formulas of such spacetimes generally filled with an anisotropic fluid with heat flow in Sec. III. Then, in Sec. IV we study the vacuum solutions, and express them all in terms of the gauge field A. Although the solutions are not unique, when we apply them to solar system tests in Sec. V, we find that these tests seemingly pick up the Schwarzschild solution generically. It should be noted that solar system tests were studied by several authors in the framework of the HL theory mainly without the projectability condition [13]. In Sec. VI, we study perfect fluid solutions without heat flow, and also express them all in terms of the gauge field A. By properly choosing A, non-singular stars can be constructed, due to the coupling of the gauge field with the fluid. This is in contrast to all the previous versions of the HL theory, in which it was shown that non-singular static perfect fluid soltuons without heat flow do not exist [14], although the ones with heat fluid do [16]. Then, we restrict ourselves to the cases where the pressure is a constant, which is quite similar to the Schwarzshcild fluid solution. We show explicitly that these solutions are free of spacetime singularities at the center of the star. In Sec. VII we consider the junction conditions across the surface of a compact object with the minimal requirement that the matching is mathematically meaningful. In particular, this allows the existence of a thin matter shell on the surface of the star, where the pressures of the thin shell can have high-order derivatives of the Dirac delta function. applying these conditions to the solutions found in Secs. V and VI, we obtain the matching conditions with or without a thin matter shell. Finally, in Sec. VIII we present our main results and concluding remarks.

It should be noted that spherically symetric static spacetimes in other versions of the HL theory have been extensively studied, and it is difficult to give a complete list here. Instead, we simply refer readers to references given in [3], and to the more recent review articles [5, 9].

II. NONRELATIVISITC GENERAL COVARIANT HL THEORY

In this section, we shall give a very brief introduction to the non-relativistic general covariant theory of gravity. For details, we refer readers to [10]. We shall closely follow [15], so that the notations and conversations will be used directly without further explanations. In the following, [15] will be referred to as Paper I.

The total action is given by,

$$\begin{aligned} S &= \zeta^2 \int dt d^3x N \sqrt{g} \Big(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A \\ &\qquad + \frac{1}{\zeta^2} \mathcal{L}_M \Big), \end{aligned} \tag{2.1}$$

where $g = \det g_{ij}$, and

$$\begin{aligned} \mathcal{L}_K &= K_{ij} K^{ij} - K^2, \\ \mathcal{L}_\varphi &= \varphi \mathcal{G}^{ij} \Big(2K_{ij} + \nabla_i \nabla_j \varphi \Big), \\ \mathcal{L}_A &= \frac{A}{N} \Big(2\Lambda_g - R \Big). \end{aligned} \tag{2.2}$$

Here the coupling constant Λ_g, acting like a 3-dimensional cosmological constant, has the dimension of $(\text{length})^{-2}$. The Ricci and Riemann terms all refer to the three-metric g_{ij}. K_{ij} is the extrinsic curvature, and $\mathcal{G}_{ij}$ is the 3-dimensional "generalized" Einstein tensor, defined by

$$\begin{aligned} K_{ij} &= \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i \right), \\ \mathcal{G}_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}. \end{aligned} \tag{2.3}$$

$\mathcal{L}_M$ is the matter Lagrangian density and $\mathcal{L}_V$ an arbitrary Diff(Σ)-invariant local scalar functional built out of the spatial metric, its Riemann tensor and spatial covariant derivatives, without the use of time derivatives. In [6], by assuming that the highest order derivatives are six and that the theory respects the parity, SVW constructed the most general form of $\mathcal{L}_V$, given by

$$\begin{aligned} \mathcal{L}_V &= \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} \left(g_2 R^2 + g_3 R_{ij} R^{ij} \right) \\ &\quad + \frac{1}{\zeta^4} \left(g_4 R^3 + g_5 R\ R_{ij} R^{ij} + g_6 R^i_j R^j_k R^k_i \right) \\ &\quad + \frac{1}{\zeta^4} \left[g_7 R \nabla^2 R + g_8 \left(\nabla_i R_{jk} \right) \left(\nabla^i R^{jk} \right) \right], \end{aligned} \tag{2.4}$$

where the coupling constants $g_s\,(s=0,1,2,\dots 8)$ are all dimensionless. The relativistic limit in the IR requires [6],

$$g_1 = -1, \quad \zeta^2 = \frac{1}{16\pi G}. \tag{2.5}$$

In Paper I, this possibility was left open. To compare with the results obtained in [16], which will be referred to as Paper II, we shall restrict ourselves to these values.

Variation of the total action (2.1) with respect to the lapse function $N(t)$ yields the Hamiltonian constraint,

$$\int d^3x\sqrt{g}\left(\mathcal{L}_K + \mathcal{L}_V - \varphi\mathcal{G}^{ij}\nabla_i\nabla_j\varphi\right) = 8\pi G\int d^3x\sqrt{g}\, J^t, \tag{2.6}$$

where

$$J^t = 2\frac{\delta\left(N\mathcal{L}_M\right)}{\delta N}. \tag{2.7}$$

Variation of the action with respect to the shift N^i yields the super-momentum constraint,

$$\nabla_j\left(\pi^{ij} - \varphi\mathcal{G}^{ij}\right) = 8\pi G J^i, \tag{2.8}$$

where the super-momentum π^{ij} and matter current J^i are defined as

$$\begin{aligned}\pi^{ij} &\equiv \frac{\delta\mathcal{L}_K}{\delta\dot{g}_{ij}} = -K^{ij} + Kg^{ij},\\ J^i &\equiv -N\frac{\delta\mathcal{L}_M}{\delta N_i}.\end{aligned} \tag{2.9}$$

Similarly, variations of the action with respect to φ and A yield,

$$\mathcal{G}^{ij}\left(K_{ij} + \nabla_i\nabla_j\varphi\right) = 8\pi G J_\varphi, \tag{2.10}$$

$$R - 2\Lambda_g = 8\pi G J_A, \tag{2.11}$$

where

$$J_\varphi \equiv -\frac{\delta\mathcal{L}_M}{\delta\varphi}, \quad J_A \equiv 2\frac{\delta\left(N\mathcal{L}_M\right)}{\delta A}. \tag{2.12}$$

On the other hand, variation with respect to g_{ij} leads to the dynamical equations,

$$\begin{aligned}\frac{1}{N\sqrt{g}}\left[\sqrt{g}\left(\pi^{ij} - \varphi\mathcal{G}^{ij}\right)\right]_{,t} &= -2\left(K^2\right)^{ij} + 2KK^{ij}\\ &+ \frac{1}{N}\nabla_k\left[N^k\pi^{ij} - 2\pi^{k(i}N^{j)}\right]\\ &+ \frac{1}{2}\left(\mathcal{L}_K + \mathcal{L}_\varphi + \mathcal{L}_A\right)g^{ij}\\ &+ F^{ij} + F_\varphi^{ij} + F_A^{ij} + 8\pi G\tau^{ij},\end{aligned} \tag{2.13}$$

where $\left(K^2\right)^{ij} \equiv K^{il}K_l^j,\ f_{(ij)} \equiv \left(f_{ij} + f_{ji}\right)/2$, and

$$\begin{aligned}F_A^{ij} &= \frac{1}{N}\left[AR^{ij} - \left(\nabla^i\nabla^j - g^{ij}\nabla^2\right)A\right],\\ F_\varphi^{ij} &= \sum_{n=1}^{3} F_{(\varphi,n)}^{ij},\\ F^{ij} &\equiv \frac{1}{\sqrt{g}}\frac{\delta\left(-\sqrt{g}\mathcal{L}_V\right)}{\delta g_{ij}} = \sum_{s=0}^{8} g_s\zeta^{n_s}\left(F_s\right)^{ij},\end{aligned} \tag{2.14}$$

with $n_s = (2,0,-2,-2,-4,-4,-4,-4,-4)$. The geometric 3-tensors $(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$ are given by Eqs.(2.21)-(2.23) in Paper I, which, for the sake of the readers' convenience, are reproduced in Eqs.(A.1) and (A.2) of this paper. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}}\frac{\delta\left(\sqrt{g}\mathcal{L}_M\right)}{\delta g_{ij}}. \tag{2.15}$$

The matter quantities $(J^t,\ J^i,\ J_\varphi,\ J_A,\ \tau^{ij})$ satisfy the conservation laws,

$$\begin{aligned}\int d^3x\sqrt{g}\Bigg[\dot{g}_{kl}\tau^{kl} - \frac{1}{\sqrt{g}}\left(\sqrt{g}J^t\right)_{,t} + \frac{2N_k}{N\sqrt{g}}\left(\sqrt{g}J^k\right)_{,t}\\ -2\dot{\varphi}J_\varphi - \frac{A}{N\sqrt{g}}\left(\sqrt{g}J_A\right)_{,t}\Bigg] = 0,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\nabla^k\tau_{ik} - \frac{1}{N\sqrt{g}}\left(\sqrt{g}J_i\right)_{,t} - \frac{J^k}{N}\left(\nabla_k N_i - \nabla_i N_k\right)\\ -\frac{N_i}{N}\nabla_k J^k + J_\varphi\nabla_i\varphi - \frac{J_A}{2N}\nabla_i A = 0.\end{aligned} \tag{2.17}$$

III. SPHERICALLY SYMMETRIC STATIC SPACETIMES

Spherically symmetric static spacetimes in the framework of the SVW setup are studied systematically in Paper II. In this section, we shall closely follow the development presented there. In particular, the metric for static spherically symmetric spacetimes that preserve the form of Eq. (1.2) with the projectability condition can be cast in the form,

$$ds^2 = -c^2dt^2 + e^{2\nu}\left(dr + e^{\mu-\nu}dt\right)^2 + r^2d^2\Omega, \tag{3.1}$$

in the spherical coordinates $x^i = (r,\theta,\phi)$, where $d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2$, and

$$\mu = \mu(r), \quad \nu = \nu(r), \quad N^i = \left\{e^{\mu-\nu}, 0, 0\right\}. \tag{3.2}$$

The corresponding timelike Killing vector is $\xi = \partial_t$. For the above metric, one finds

$$\begin{aligned}K_{ij} &= e^{\mu+\nu}\Big(\mu'\delta_i^r\delta_j^r + re^{-2\nu}\Omega_{ij}\Big),\\ R_{ij} &= \frac{2\nu'}{r}\delta_i^r\delta_j^r + e^{-2\nu}\Big[r\nu' - \left(1 - e^{2\nu}\right)\Big]\Omega_{ij},\\ \mathcal{L}_K &= -\frac{2}{r^2}e^{2(\mu-\nu)}\left(2r\mu' + 1\right),\end{aligned}$$

$$\mathcal{L}_\varphi = \frac{\varphi e^{-4\nu}}{r^2}\Bigg\{\Big[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\Big]\Big(\varphi'' - \nu'\varphi' + 2e^{\mu+\nu}\mu'\Big) - 2\Big(\nu' - \Lambda_g r e^{2\nu}\Big)\Big(\varphi' + 2e^{\mu+\nu}\Big)\Bigg\},$$

$$\mathcal{L}_A = \frac{2A}{r^2}\Big[e^{-2\nu}\left(1 - 2r\nu'\right) + \left(\Lambda_g r^2 - 1\right)\Big],$$

$$\mathcal{L}_V = \sum_{s=0}^{3}\mathcal{L}_V^{(s)}, \tag{3.3}$$

where a prime denotes the ordinary derivative with respect to its indicated argument, $\Omega_{ij} \equiv \delta^\theta_i\delta^\theta_j + \sin^2\theta\delta^\phi_i\delta^\phi_j$, and $\mathcal{L}_V^{(s)}$'s are given by Eq.(A1) in Paper II. Then, the Hamiltonian constraint (2.6) reads,

$$\int\Big(\mathcal{L}_K + \mathcal{L}_V - \mathcal{L}_\varphi^{(1)} - 8\pi G J^t\Big)e^\nu r^2 dr = 0, \tag{3.4}$$

where

$$\mathcal{L}_\varphi^{(1)} = \frac{\varphi e^{-4\nu}}{r^2}\Bigg\{\Big[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\Big](\varphi'' - \nu'\varphi') - 2\left(\nu' - \Lambda_g r e^{2\nu}\right)\varphi'\Bigg\}, \tag{3.5}$$

while the momentum constraint (2.8) yields,

$$2r\nu' + e^{-(\mu+\nu)}\Big[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\Big]\varphi' = -8\pi G r^2 e^{2(\nu-\mu)} v, \tag{3.6}$$

where $J^i = e^{-(\mu+\nu)}\big(v, 0, 0\big)$. It can be also shown that Eqs.(2.10) and (2.11) now read,

$$\Big[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\Big]\Big(\varphi'' - \nu'\varphi' + e^{\mu+\nu}\mu'\Big) - 2\Big(\nu' - \Lambda_g r e^{2\nu}\Big)\Big(\varphi' + e^{\mu+\nu}\Big) = 8\pi G r^2 e^{4\nu} J_\varphi, \tag{3.7}$$

$$2r\nu' - \Big[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\Big] = 4\pi G r^2 e^{2\nu} J_A. \tag{3.8}$$

The dynamical equations (2.13), on the other hand, yield,

$$2\big(\mu' + \nu'\big) + \frac{1}{r} + \frac{1}{2} r e^{2(\nu-\mu)}\left(\mathcal{L}_\varphi + \mathcal{L}_A\right) = -r e^{-2\mu}\Big(F_{rr} + F^\varphi_{rr} + F^A_{rr} + 8\pi G e^{2\nu} p_r\Big), \tag{3.9}$$

$$\mu'' + \big(2\mu' - \nu'\big)\left(\mu' + \frac{1}{r}\right) + \frac{1}{2}e^{2(\nu-\mu)}\left(\mathcal{L}_\varphi + \mathcal{L}_A\right) = -\frac{e^{2(\nu-\mu)}}{r^2}\left(F_{\theta\theta} + F^\varphi_{\theta\theta} + F^A_{\theta\theta} + 8\pi G r^2 p_\theta\right), \tag{3.10}$$

where

$$\tau_{ij} = e^{2\nu} p_r \delta^r_i\delta^r_j + r^2 p_\theta \Omega_{ij},$$

$$F^A_{ij} = \frac{2}{r}\left(A' + A\nu'\right)\delta^r_i\delta^r_j + e^{-2\nu}\Big[r^2\left(A'' - \nu' A'\right) + r\left(A' + A\nu'\right) - A\Big(1 - e^{2\nu}\Big)\Big]\Omega_{ij}, \tag{3.11}$$

F_{ij}'s for the metric (3.1) are given by Eq.(A2) in Paper II, and $F^{ij}_{(\varphi,s)}$ are given by Eq.(B.1) in the present paper. As in Paper II, here we define a fluid with $p_r = p_\theta$ as a perfect fluid, which in general conducts heat flow along the radial direction [17].

Since the spacetime is static, one can see that now the energy conservation law (2.16) is satisfied identically, while the momentum conservation (2.17) yields,

$$v\mu' - \big(v' - p_r'\big) - \frac{2}{r}\big(v - p_r + p_\theta\big) + J_\varphi\varphi' - \frac{1}{2}J_A A' = 0. \tag{3.12}$$

To relate the quantities J^t, J^i and τ_{ij} to the ones often used in general relativity, following Paper II, one can first introduce the unit normal vector n_μ to the hypersurfaces $t =$ Constant, and then the spacelike unit vectors χ_μ, θ_μ and ϕ_μ, defined as [18]

$$n_\mu = \delta^t_\mu, \quad n^\mu = -\delta^\mu_t + e^{\mu-\nu}\delta^\mu_r,$$

$$\chi^\mu = e^{-\nu}\delta^\mu_r, \quad \chi_\mu = e^\mu\delta^t_\mu + e^\nu\delta^r_\mu,$$

$$\theta_\mu = r\delta^\theta_\mu, \quad \phi_\mu = r\sin\theta\delta^\phi_\mu. \tag{3.13}$$

In terms of these four unit vectors, the energy-momentum tensor for an anisotropic fluid with heat flow can be written as

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + q\big(n_\mu\chi_\nu + n_\nu\chi_\mu\big) + p_r\chi_\mu\chi_\nu + p_\theta\big(\theta_\mu\theta_\nu + \phi_\mu\phi_\nu\big), \tag{3.14}$$

where ρ_H, q, p_r and p_θ denote, respectively, the energy density, heat flow along radial direction, radial, and tangential pressures, measured by the observer with the four-velocity n_μ. Then, one can see that such a decomposition is consistent with the quantities J^t and J^i, defined by

$$\rho_H = -2J^t, \quad v = e^\mu q. \tag{3.15}$$

It should be noted that the definitions of the energy density ρ_H, the radial pressure p_r and the heat flow q are different from the ones defined in a comoving frame in general relativity. For detail, we refer readers to Appendix B of Paper II.

Finally, we note that in writing the above equations, we leave the choice of the $U(1)$ gauge open. From Eq.(1.9) one can see that it can be used to set one (and only one) of the three functions A, φ and N_r to zero. To compare our results with the one obtained in [10], in the rest of this paper (except the first part of Sec. VII), without loss of the generality, we shall choose the gauge,

$$\varphi = 0. \tag{3.16}$$

Then, we find that

$$\mathcal{L}_\varphi = 0, \quad F^{ij}_{(\varphi,n)} = 0, \; (n = 1, 2, 3). \tag{3.17}$$

IV. VACUUM SOLUTIONS

In the vacuum case, we have $J^t = v = p_r = p_\theta = J_A = J_\varphi = 0$. With the gauge (3.16), from the momentum constraint (3.6) we immediately obtain $\nu =$ Constant, while Eq. (3.8) further requires

$$\nu = 0, \quad \Lambda_g = 0. \tag{4.1}$$

This is different from the solutions presented in [10], where $\nu \neq 0$, $\mu = -\infty$. Inserting the above into Eq.(3.7), it can be shown that it is satisfied identically. Since $\nu = 0$, from the expressions of $(F_s)_{ij}$ given by Eq.(A2) in Paper II, we find that $(F_s)_{ij} = 0$ for $s \neq 0$, and $(F_0)_{ij} = -g_{ij}/2$, so that

$$F_{ij} = -\Lambda g_{ij}, \tag{4.2}$$

where $\Lambda \equiv \zeta^2 g_0/2$. Substituting Eqs.(3.11) and (3.17) - (4.2) into Eqs.(3.9) and (3.10), we find that

$$\Big(2r\mu' + 1\Big)e^{2\mu} = \Lambda r^2 - 2rA', \tag{4.3}$$

$$\mu'' + 2\mu'\Big(\mu' + \frac{1}{r}\Big) = \frac{e^{-2\mu}}{r}\Big[\Lambda r - \big(rA'\big)'\Big]. \tag{4.4}$$

It can be shown that Eq.(4.4) is not independent, and can be obtained from Eq.(4.3). Therefore, the solutions are not uniquely determined, since now we have only one equation, (4.3), for two unknowns, μ and A. In particular, for any given A, from Eq.(4.3) we find that

$$\mu = \frac{1}{2}\ln\left[\frac{2m}{r} + \frac{1}{3}\Lambda r^2 - 2A(r) + \frac{2}{r}\int^r A(r')dr'\right]. \tag{4.5}$$

On the other hand, we also have

$$\mathcal{L}_K = \frac{4A'}{r} - 2\Lambda, \quad \mathcal{L}_V = 2\Lambda. \tag{4.6}$$

Inserting it into the Hamiltonian constraint (3.4), we find that

$$\int_0^\infty rA'(r)dr = 0. \tag{4.7}$$

Therefore, for any given function A, subjected to the above constraint, the solutions given by Eqs.(4.1) and (4.5) represent the vacuum solutions of the HL theory. Thus, in contrast to general relativity, the vacuum solutions in the HMT setup are not unique.

When A is a constant (without loss of generality, we can set $A = 0$), from the above we find that

$$\mu = \frac{1}{2}\ln\left(\frac{2m}{r} + \frac{1}{3}\Lambda r^2\right), \ (A = 0), \tag{4.8}$$

which is exactly the Schwarzschild (anti-) de Sitter solution, written in the Gullstrand-Painleve coordinates [19]. It is interesting to note that when $m = 0$ we must assume that $\Lambda > 0$, in order to have μ real. That is, the anti-de Sitter solution cannot be written in the static ADM form (3.1).

V. SOLAR SYSTEM TESTS

The solar system tests are usually written in terms of the Eddington parameters, by following the so-called "parameterized post-Newtonian" (PPN) approach, introduced initially by Eddington [20]. The gravitational field, produced by a point-like and motion-less particle with mass M, is often described by the form of metric [21],

$$ds^2 = -e^{2\Psi}c^2dt^2 + e^{2\Phi}dr^2 + r^2d^2\Omega, \tag{5.1}$$

where Ψ and Φ are functions of the dimensionless quantity $\chi \equiv GM/(rc^2)$ only. For the solar system, we have $GM_{\bigodot}/c^2 \simeq 1.5km$, so that in most cases we have $\chi \ll 1$. Expanding Ψ and Φ in terms of χ, we have [21]

$$\begin{aligned} e^{2\Psi} &= 1 - 2\left(\frac{GM}{c^2r}\right) + 2\big(\beta - \gamma\big)\left(\frac{GM}{c^2r}\right)^2 + ..., \\ e^{2\Phi} &= 1 + 2\gamma\left(\frac{GM}{c^2r}\right) + ..., \end{aligned} \tag{5.2}$$

where β and γ are the Eddington parameters. General relativity predicts $\beta = 1 = \gamma$ strictly, while the current radar ranging of the Cassini probe [22], and the procession of lunar laser ranging data [23] yield, respectively, the bounds [24],

$$\begin{aligned} \gamma - 1 &= (2.1 \pm 2.3) \times 10^{-5}, \\ \beta - 1 &= (1.2 \pm 1.1) \times 10^{-4}, \end{aligned} \tag{5.3}$$

which are consistent with the predictions of general relativity.

To apply the solar system tests to the HL theory, we need first to transfer the above bounds to the metric coefficients μ and ν. In Appendix B of Paper II, the relations between (Φ, Ψ) and (μ, ν) have been worked out explicitly, and are given by

$$\mu = \frac{1}{2}\ln\left[c^2\Big(1 - e^{2\Psi}\Big)\right], \quad \nu = \Phi + \Psi, \tag{5.4}$$

or inversely,

$$\begin{aligned} \Phi &= \nu - \frac{1}{2}\ln\Big(1 - \frac{1}{c^2}e^{2\mu}\Big), \\ \Psi &= \frac{1}{2}\ln\Big(1 - \frac{1}{c^2}e^{2\mu}\Big). \end{aligned} \tag{5.5}$$

Inserting Eq.(5.2) into Eq.(5.4), we find that

$$\begin{aligned} \mu &= \frac{1}{2}\ln\left\{2c^2\left[\left(\frac{GM}{c^2r}\right) - (\beta - \gamma)\left(\frac{GM}{c^2r}\right)^2 + ...\right]\right\}, \\ \nu &= \big(\gamma - 1\big)\left(\frac{GM}{c^2r}\right) + ... \end{aligned} \tag{5.6}$$

Comparing Eq.(5.6) with Eq.(4.5) for $\Lambda = 0$, we find that in order to be consistent with solar system tests, we must assume that

$$A(r) = \mathcal{O}\left[\left(\frac{GM}{c^2r}\right)^2\right]. \tag{5.7}$$

Together with the Hamiltonian constraint (4.7), we find that this is impossible unless $A = 0$. Therefore, although the vacuum solution in the HMT setup is not unique, the solar system tests seemingly require that it must be the Schwarzschild vacuum solution.

It should be noted that by choosing $A(r)$ in very particular forms, the condition $A = 0$ could be relaxed [25]. But, such chosen A is not analytic (in terms of the dimensionless quantity χ), and it is not clear how to expand it in the form of (5.6). Thus, in this paper we simply discard those possibilities.

VI. PERFECT FLUID SOLUTIONS

In this section, let us consider perfect fluid without heat flow, that is,

$$p_r = p_\theta = p, \quad v = 0. \tag{6.1}$$

Then, together with the gauge choice (3.16), from Eq.(3.6) we find that $\nu =$ Constant. However, to be matched with the vacuum solutions outside of the fluid, as shown in Sec. IV, we must set this constant to zero,

$$\nu = 0, \tag{6.2}$$

from which we immediately find that $R_{ij} = 0$, F_{ij} is still given by Eq.(4.2), and

$$\begin{aligned} \mathcal{L}_A &= 2\Lambda_g A, \\ F^A_{ij} &= \frac{2A'}{r}\delta^r_i\delta^r_j + r\big(rA'\big)'\Omega_{ij}. \end{aligned} \tag{6.3}$$

Inserting the above into Eqs.(3.7)-(3.10), we find that

$$J_\varphi = \frac{\Lambda_g}{8\pi G r^2}\left(r^2 e^\mu\right)_{,r}, \tag{6.4}$$

$$J_A = -\frac{\Lambda_g}{4\pi G}, \tag{6.5}$$

$$\big(rf\big)' + 2rA' + \Lambda_g r^2 A - \Lambda r^2 = -8\pi G r^2 p, \tag{6.6}$$

$$\frac{1}{2}rf'' + f' + \big(rA'\big)' + \Lambda_g rA - \Lambda r = -8\pi G r p, \tag{6.7}$$

where $f \equiv e^{2\mu}$. From the last two equations, we find that

$$r^2 f'' - 2f = -2r^3\left(\frac{A'}{r}\right)'. \tag{6.8}$$

On the other hand, the conservation law of momentum (3.12) now reduces to

$$p' + \frac{\Lambda_g}{8\pi G}A' = 0, \tag{6.9}$$

which has the solution,

$$p = p_0 - \frac{\Lambda_g}{8\pi G}A, \tag{6.10}$$

where p_0 is an integration constant.

Substituting it into Eq.(6.6), and then taking a derivative of it, we find that the resulting equation is exactly given by Eq.(6.8). Thus, both Eqs.(6.8) and (6.7) are not independent, and can all be derived from Eqs.(6.6) and (6.10). Then, in the present case there are five independent equations, the Hamiltonian constraint (3.4), and Eqs.(6.4), (6.5), (6.6) and (6.10). However, we have six unknowns, A, μ, p, J^t, J_φ, J_A. Therefore, the problem now is not uniquely determined. As in the vacuum case, we can express all these quantities in terms of the gauge field A. In particular, substituting Eq.(6.10) into Eq.(6.6) and then integrating it, we obtain

$$\mu = \frac{1}{2}\ln\left[\frac{2m}{r} + \frac{1}{3}\Big(\Lambda - 8\pi G p_0\Big)r^2 - 2A + \frac{2}{r}\int^r A(r')dr'\right]. \tag{6.11}$$

Then, we find that

$$\mathcal{L}_\varphi = \frac{4A'}{r} - 2\left(\Lambda - 8\pi G p_0\right), \quad \mathcal{L}_V = 2\Lambda. \tag{6.12}$$

Inserting the above into Eq.(3.4), we find that it can be cast in the form,

$$\int_0^\infty \tilde{\rho}(r)dr = 0, \tag{6.13}$$

where

$$J^t = \frac{1}{2\pi G}\left(4\pi G p_0 + \frac{A'(r)}{r} - \frac{\tilde{\rho}(r)}{r^2}\right). \tag{6.14}$$

From the above one can see that once A is given, one can immediately obtain all the rest. By properly choosing it (and $\tilde{\rho}(r)$), it is not difficult to see that one can construct non-singular solutions representing stars made of a perfect fluid without heat flow. To see this explicitly, let us consider the following two particular cases.

A. $\Lambda_g = 0$

When $\Lambda_g = 0$, we have

$$J_\varphi = J_A = 0, \quad p = p_0, \tag{6.15}$$

while μ and J^t are still given by Eqs.(6.11) and (6.14), respectively. To have a physically acceptable model, we require that the fluid be non-singular in the center. Since $R_{ij} = 0$, one can see that any quantity built from the Riemann and Ricci tensor vanishes in the present case. Then, possible singularities can only come from the kinetic part, K_{ij}, where the very first quantity is

$$\begin{aligned} K &= g^{ij}K_{ij} = \frac{e^\mu}{r}\left(r\mu' + 2\right) \\ &= \frac{e^{-\mu}}{r}\left(\frac{3m}{r} + \left(\Lambda - 8\pi G p_0\right)r^2 - rA' - 3A \right. \\ &\qquad \left. + \frac{3}{r}\int A(r')dr'\right). \end{aligned} \tag{6.16}$$

Assuming that near the center A is dominated by the term r^{α}, we find that K is non-singular only when

$$m = 0, \quad \alpha \geq 2. \tag{6.17}$$

For such a function A, Eq.(6.14) show that J^t is non-simgular, as long as $\tilde{\rho}(r) \simeq \mathcal{O}(r^2)$.

B. $A = A_0$

When A is a constant, from Eq.(6.10) we can see that the pressure p is also a constant. Then, the integration of Eq.(6.6) yields,

$$\mu = \frac{1}{2}\ln\left\{\frac{2m}{r} + \frac{1}{3}\Big(\Lambda - 8\pi G p_0\Big)r^2\right\}. \tag{6.18}$$

Inserting it into Eq.(6.8) we find that it is satisfied identically, while the Hamiltonian constraint (3.4) can also be cast in the form of Eq.(6.13), but now with

$$J^t = \frac{1}{8\pi G}\left(16\pi G p_0 - \frac{\tilde{\rho}(r)}{r^2}\right). \tag{6.19}$$

Thus, the solutions of Eqs.(6.2) and (6.18) represent a perfect fluid with a constant pressure, $p = p(A_0)$. In this case, it can be shown that K is free of any spacetime singularity at the center only when $m = 0$.

It should be noted that in [14] it was shown that non-singular static solutions of perfect fluid without heat flow do not exist. Since their conclusions only come from the conservation law of momentum, one might expect that this is also true in the current setup. However, from Eq.(3.12) we can see that in the present case the conservation law contains two extra terms, J_{φ} and J_A. Only when both of them vanish, can one obtain the above conclusions. Since in general one can only choose one of them to be zero by using the gauge freedom, it is expected that non-singular static stars can be constrcuted by properly choosing the gauge field A.

It should be also noted that the arguments presented in [14] do not apply to the case where the pressure is a constant. Therefore, when $p' = 0$ non-singular stars without heat flow can be also constrcuted in other versions of the HL theory, although when $p' \neq 0$ this is possible only in the HMT setup. In addition, the definitions of the quantities ρ, p and v ($\equiv q e^{\mu}$) used in this paper are different from the ones used usually in general relativity. For detail, we refer readers to Appendix B of [16], specially to Eq.(B16).

VII. JUNCTION CONDITIONS

To consider the junction conditions across the hypersurface of a compact object, let us first divide the whole spacetime into three regions, $V^{\pm}$ and Σ, where V^- (V^+) denotes the internal (external) region of the star, and Σ is the surface of the star. As shown in Paper II, once the metric is cast in the form (3.1), the coordinates t and r are all uniquely defined, so that the coordinates defined in V^+ and V^- must be the same, $\{x^{+\mu}\} = \{x^{-\mu}\} = (t, r, \theta, \phi)$. Since the quadratic terms of the highest derivatives of the metric coefficients μ and ν are only terms of the forms, ν''^2, $\nu''\nu'''$ and μ'^2, the minimal requirements for these two functions are that $\nu(r)$ and $\mu(r)$ are respectively at least C^1 and C^0 across the surface Σ, and that they are at least C^4 and C^1 elsewhere. For detail, we refer readers to [16]. Similarly, the quadratic terms of the Newtonian pre-potential are only involved with the forms, $\varphi\varphi''$, $\varphi_{,r}^2$, and $\varphi\varphi'$. Therefore, the minimal requirment for φ is to be at least C^0 across the surface Σ. On the other hand, the gauge field A and its derivatives all appear linearly. Thus, mathematically it can be even not continuous across Σ. However, in this paper we shall require that A be at least C^0 too across Σ. Elsewhere, A and φ are at least C^1. Then, we can write A and φ in the form,

$$E(r) = E^+(r)H\left(r - r_0\right) + E^-(r)\left[1 - H\left(r - r_0\right)\right], \tag{7.1}$$

where $E = (A, \varphi)$, r_0 is the radius of the star, and $H(x)$ denotes the Heavside function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{7.2}$$

Since A and φ are continuous (C^0) across $r = r_0$, we must have

$$\text{limit}_{r\to r_0^+} E^+(r) = \text{limit}_{r\to r_0^-} E^-(r). \tag{7.3}$$

Then, we find that

$$\begin{aligned} E'(r) &= E^D_{,r}(r), \\ E''(r) &= E^D_{,rr}(r) + \left[E'\right]^- \delta\left(r - r_0\right), \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} \left[E'\right]^- &\equiv \text{limit}_{r\to r_0^+} E^+_{,r}(r) - \text{limit}_{r\to r_0^-} E^-_{,r}(r), \\ E^D(r) &\equiv E^+ H(r - r_0) + E^-\Big[1 - H(r - r_0)\Big]. \end{aligned} \tag{7.5}$$

Combining the above with Eq.(6.6) of Paper II, we find that

$$\begin{aligned} \mathcal{L}_K &= \mathcal{L}_K^D, \quad \mathcal{L}_A = \mathcal{L}_A^D, \\ \mathcal{L}_V &= \mathcal{L}_V^D + \mathcal{L}_V^{Im}\delta(r - r_0), \\ \mathcal{L}_{\varphi} &= \mathcal{L}_{\varphi}^D + \mathcal{L}_{\varphi}^{Im}\delta(r - r_0), \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} \mathcal{L}_V^{Im} &\equiv \frac{8 g_7 e^{-6\nu}}{\zeta^4 r^3}\Big[2r\nu' - \left(1 - e^{2\nu}\right)\Big]\left[\nu''\right]^-, \\ \mathcal{L}_{\varphi}^{Im} &\equiv \frac{\varphi e^{-4\nu}}{r^2}\Big[\Lambda_g r^2 e^{2\nu} + \left(1 - e^{2\nu}\right)\Big]\left[\varphi'\right]. \end{aligned} \tag{7.7}$$

Setting

$$J = J^D + J^{Im}\delta(r - r_0), \tag{7.8}$$

where $J \equiv \{J^t,\ v,\ J_\varphi,\ J_A\}$, and J^{Im} has support only on Σ, we find that the Hamiltonian constraint (3.4) can be written as

$$\int^D \left(\mathcal{L}_K + \mathcal{L}_V - \mathcal{L}_\varphi^{(1)} - 8\pi G J^t\right) e^\nu r^2 dr = \frac{1}{4\pi}\Big(8\pi G J^{t,Im} + \mathcal{L}_\varphi^{Im} - \mathcal{L}_V^{Im}\Big), \tag{7.9}$$

where

$$\int^D I(r)dr = \text{limit}_{\epsilon\to 0}\left(\int_0^{r_0-\epsilon} I(r)dr + \int_{r_0+\epsilon}^{\infty} I(r)dr\right). \tag{7.10}$$

It shold be noted that in writing Eq.(7.9), we had used the conversion,

$$\int \sqrt{g} d^3x f(r)\delta(r - r_0) = f(r_0). \tag{7.11}$$

The momentum constraint (3.6) will take the same form in Regions $V^\pm$, while on the surface Σ it yields

$$v^{Im} = 0. \tag{7.12}$$

That is, the surface does not support impulsive heat flow in the radial direction. Similarly, Eqs.(3.7) and (3.8) take the same forms in Regions $V^\pm$. While on Σ they reduce, respectively, to

$$\Big[\Lambda_g r^2 e^{2\nu} + \big(1 - e^{2\nu}\big)\Big]\left[\varphi'\right]^- = 8\pi G r^2 e^{4\nu} J_\varphi^{Im}, \tag{7.13}$$

$$J_A^{Im} = 0,\ (r = r_0). \tag{7.14}$$

On the other hand, the dynamical equations (3.9) and (3.10) take the same forms in Regions $V^\pm$, and on the surface Σ they yield,

$$\begin{aligned}
&\Bigg\{\varphi e^{-2\nu}\Big[\Lambda_g r^2 e^{2\nu} + \big(1 - e^{2\nu}\big)\Big]\left[\varphi'\right]^- \\
&\quad + 2r^2 F_{rr}^{\varphi,Im}\Bigg\}\delta(r - r_0) \\
&= -2r^2\Big(F_{rr}^{Im} + 8\pi G e^{2\nu} p_r^{Im}\Big),
\end{aligned} \tag{7.15}$$

$$\begin{aligned}
&\Bigg\{\left[\mu'\right]^- + \frac{1}{2}e^{2(\nu-\mu)}\mathcal{L}_\varphi^{Im} \\
&\quad + \frac{e^{2(\nu-\mu)}}{r^2}\Big(F_{\theta\theta}^{\varphi,Im} + F_{\theta\theta}^{A,Im}\Big)\Bigg\}\delta(r - r_0) \\
&= -\frac{e^{2(\nu-\mu)}}{r^2}\Big(F_{\theta\theta}^{Im} + 8\pi G r^2 p_\theta^{Im}\Big),
\end{aligned} \tag{7.16}$$

where F_{ij}^{Im} are given by Eq.(A5) in Paper II, $F_{ij}^{\varphi,Im}$ are given by Eq.(C.2) in Appendix C of this paper, and

$$\begin{aligned}
F_{ij}^{A,Im} &= r^2 e^{-2\nu}\left[A'\right]^- \Omega_{ij}, \\
L &= L^D + L^{Im},
\end{aligned} \tag{7.17}$$

with $L \equiv (p_r,\ p_\theta)$. From Eq.(A5) in Paper II we can see that L^{Im} in general takes the form,

$$L^{Im} = L^{(0)Im}\delta(r-r_0) + L^{(1)Im}\delta'(r-r_0) + L^{(2)Im}\delta''(r-r_0). \tag{7.18}$$

The above represents the general junction conditions of a spherical compact object made of a fluid with heat flow, in which a thin matter shell appears on Σ.

In the following we shall consider the matching of the perfect fluid solutions found in Sec. VI to the vacuum ones found in Sec. V. Since

$$\nu^\pm = 0, \quad \varphi^\pm = 0, \tag{7.19}$$

we immediately obtain

$$\begin{aligned}
R_{ij}^\pm &= 0, \quad \mathcal{L}_V^\pm = 2\Lambda_\pm, \quad \mathcal{L}_V^{Im} = 0, \\
F_{ij}^\pm &= -\Lambda_\pm g_{ij}^\pm, \quad F_{ij}^{Im} = 0, \\
\mathcal{L}_\varphi^\pm &= \mathcal{L}_\varphi^{Im} = 0, \quad \big(F_\varphi^\pm\big)_{ij} = \big(F_\varphi^{Im}\big)_{ij} = 0.
\end{aligned} \tag{7.20}$$

Then, from Eqs.(7.12), (7.13), (7.14) and (7.15) we find

$$v^{Im} = J_\varphi^{Im} = J_A^{Im} = p_r^{Im} = 0. \tag{7.21}$$

That is, the radial pressure of the thin shell must vanish. This is similar to what happened in the relativistic case [17].

In the external region, V^+, the spacetime is vacuum, and the general solutions are given by Eq.(4.5),

$$\mu^+ = \frac{1}{2}\ln\left[\frac{2m}{r} + \frac{1}{3}\Lambda_+ r^2 - 2A^+(r) + \frac{2}{r}\int^r A^+(r')dr'\right], \tag{7.22}$$

for which we have

$$\mathcal{L}_K^+ = \frac{4A_{,r}^+}{r} - 2\Lambda_+, \quad \mathcal{L}_A^+ = 0. \tag{7.23}$$

In the internal region, V^-, two classes of solutions of perfect fluid without heat flow are found, given, respectively, by Eqs.(6.11) and (6.18), which can be written as,

$$\begin{aligned}
\mu^- &= \frac{1}{2}\ln\Bigg\{\frac{1}{3}\Big(\Lambda_- - 8\pi G p_0\Big)r^2 - 2A^-(r) \\
&\qquad + \frac{2}{r}\int^r A^-(r')dr'\Bigg\},
\end{aligned} \tag{7.24}$$

for $\Lambda_g^- = 0$, and

$$\mu^- = \frac{1}{2}\ln\left\{\frac{1}{3}\Big[\Lambda_- - \Lambda_g^- A_0 - 8\pi G p\Big]r^2\right\}, \tag{7.25}$$

for $A^- = A_0$, where $p \equiv \Lambda_g^- A_0^2/2 + p_0$. Then, we find that

$$\begin{aligned}
\mathcal{L}_K^- &= \begin{cases} 2\big(8\pi G p_0 - \Lambda_-\big) + \frac{4A_{,r}^-}{r}, & \Lambda_g^- = 0, \\ 2\big(8\pi G p - \Lambda_- + \Lambda_g^- A_0\big), & A^- = A_0, \end{cases} \\
\mathcal{L}_A^- &= 2\Lambda_g^- A^-(r).
\end{aligned} \tag{7.26}$$

To further study the junction conditions, let us consider the two cases $\Lambda_g^- = 0$ and $A^- = A_0$ separately.

A. $\Lambda_g^- = 0$

In this case, the continuity conditions of μ and A across Σ read,

$$m + \frac{1}{6}\Delta\Lambda r_0^3 + \int^{r_0} \Delta A(r)dr = -\frac{4\pi G}{3}p_0 r_0^3,$$
$$A^+(r_0) = A^-(r_0), \tag{7.27}$$

where $\Delta\Lambda \equiv \Lambda_+ - \Lambda_-$ and $\Delta A = A^+ - A^-$. Then, the Hamiltonian constraint (7.9) becomes,

$$\int_0^{r_0} \tilde{\rho}(r)dr + \int_{r_0}^{\infty} rA^+_{,r}(r)dr = \frac{1}{2}GJ^{t,Im}, \tag{7.28}$$

while the dynamical equation (7.16) reduces to,

$$\Delta\Lambda = -8\pi G\Big(p_0 + 2p_\theta^{(0)Im}\Big), \tag{7.29}$$

where $p_\theta^{Im} \equiv p_\theta^{(0)Im}\delta(r - r_0)$ [cf. Eq.(7.18)].

When the matter thin shell does not exist, we must set $J^{t,Im} = p_\theta^{(0)Im} = 0$, and Eqs.(7.27)-(7.29) become the matching conditions for the constants $\Lambda_\pm$, m, p_0 and the functions $A^\pm(r)$ and $\tilde{\rho}(r)$.

B. $A = A_0$

In this case, it can be shown that the continuity conditions for μ and A become,

$$m + \frac{1}{6}\Delta\Lambda r_0^3 + \int^{r_0} A^+(r)dr = r_0 A_0 - \frac{1}{6}\big(\Lambda_g^- A_0 + 8\pi G p\big)r_0^3,$$
$$A^+(r_0) = A_0, \tag{7.30}$$

while the Hamiltonian constraint (7.9) reduces to,

$$\int_0^{r_0} \tilde{\rho}(r)dr + 4\int_{r_0}^{\infty} rA^+_{,r}(r)dr = 2GJ^{t,Im}. \tag{7.31}$$

The dynamical equation (7.16), on the other hand, yields,

$$\Delta\Lambda = -\Lambda_g^- A_0 - 8\pi G\Big(p + 2p_\theta^{(0)Im}\Big). \tag{7.32}$$

In all the above cases, one can see that the matching is possible even without a thin matter shell on the surface of the star, $J^{t,Im} = 0 = p_\theta^{(0)Im}$, by properly choosing the free parameters.

VIII. CONCLUSIONS

In this paper, we have systematically studied spherically symmetric static spacetimes generally filled with an anisotropic fluid with heat flow along the radial direction. When the spacetimes are vacuum, we have found solutions, given explicitly by Eqs.(4.1) and (4.5), from which one can see that the solution is not unique, because the gauge field A is undetermined. When $A = 0$, the solutions reduce to the Schwarzschild (anti-) de Sitter solution. We have also studied the solar system tests, and found the constraint on the choice of A.

It should be noted that we have adopted a different point of view of the gauge field A in the IR limit than that adopted in [10]. In this paper we have considered it as independent from the 4D metric $g_{\mu\nu}$, although it interacts with them through the field equations. This is quite similar to the Brans-Dicke (BD) scalar field in the BD theory, where the scalar field represents a degree of freedom of gravity, is independent of the metric, and its effects to the spacetime are only through the field equations [26]. On the contrary, in [10] the authors considered the gauge field A as a part of the lapse function, $g_{tt} \simeq -(N - A)^2$ in the IR limit.

We have also investigated anisotropic fluids with heat flow, and found perfect fluid solutions, given by Eq.(6.11). By properly choosing the gauge field A, the solutions can be free of spacetime singularities at the center. This is in contrast to other versions of the HL theory [14], due to the coupling of the fluid with the gauge field. We then have considered two particular cases, in which the pressure is a constant, quite similar to the Schwarzschild perfect fluid solution. In all these cases, the spacetimes are free of singularities at the center.

For a compact object, the spacetime outside of it is vacuum, matching conditions are needed across the surface of the star. With the minimal requirement that the junctions be mathematically meaningful, we have worked out the general matching conditions, given by Eqs.(7.9) and (7.12)-(7.16), in which a thin matter shell in general appears on the surface of the star. Applying them to the perfect fluids, where the spacetime outside is described by the vacuum solutions (4.5), we have found the matching conditions in terms of the free parameters of the solutions. When the thin shell is removed, these conditions can also be satisfied by properly choosing the free parameters.

Finally, we note that da Silva argued, in the HMT setup, that the coupling constant λ can still be different from one [12]. If this is indeed the case, then one might be concerned with the strong coupling problem found in other versions of the HL theory [8, 27, 28]. However, since the spin-0 graviton is eliminated completely here, as shown explicitly in [10, 12, 15], this question is automatically solved in the HMT setup even with $\lambda \neq 1$ [29]. It should be noted that da Silva considered only perturbations of the case with detailed balance condition, and found that the spin-0 mode is not propagating. It is not clear if it is also true for the case without detailed balance. The problem certainly deserves further investigations.

Note Added: A preprint [25] appeared in arXiv almost simultaneously with ours. These authors also studied spherically symmetric static vacuum solutions similar

to those in Sec. IV, but did not consider the subjects presented in the other parts of this paper.

Acknowledgements: One of the authors (AW) would like to thank Shinji Mukohyama, Tony Padilla, and Thomas P. Sotiriou for valuable comments and suggestions. The work of AW was supported in part by DOE Grant, DE-FG02-10ER41692.

Appendix A: Functions $(F_s)_{ij}$ and $F^{ij}_{(\varphi,n)}$

The geometric 3-tensors F^{ij} and $F^{ij}_{(\varphi,n)}$ defined in Eq.(2.14) are given by

$$\begin{aligned}
(F_0)_{ij} &= -\frac{1}{2}g_{ij},\\
(F_1)_{ij} &= R_{ij}-\frac{1}{2}Rg_{ij},\\
(F_2)_{ij} &= 2\left(R_{ij}-\nabla_i\nabla_j\right)R-\frac{1}{2}g_{ij}\left(R-4\nabla^2\right)R,\\
(F_3)_{ij} &= \nabla^2 R_{ij}-\left(\nabla_i\nabla_j-3R_{ij}\right)R-4\left(R^2\right)_{ij}\\
&\quad +\frac{1}{2}g_{ij}\left(3R_{kl}R^{kl}+\nabla^2R-2R^2\right),\\
(F_4)_{ij} &= 3\left(R_{ij}-\nabla_i\nabla_j\right)R^2-\frac{1}{2}g_{ij}\left(R-6\nabla^2\right)R^2,\\
(F_5)_{ij} &= \left(R_{ij}+\nabla_i\nabla_j\right)\left(R_{kl}R^{kl}\right)+2R\left(R^2\right)_{ij}\\
&\quad +\nabla^2\left(RR_{ij}\right)-\nabla^k\left[\nabla_i\left(RR_{jk}\right)+\nabla_j\left(RR_{ik}\right)\right]\\
&\quad -\frac{1}{2}g_{ij}\left[\left(R-2\nabla^2\right)\left(R_{kl}R^{kl}\right)\right.\\
&\quad \left.-2\nabla_k\nabla_l\left(RR^{kl}\right)\right],\\
(F_6)_{ij} &= 3\left(R^3\right)_{ij}+\frac{3}{2}\left[\nabla^2\left(R^2\right)_{ij}\right.\\
&\quad \left.-\nabla^k\left(\nabla_i\left(R^2\right)_{jk}+\nabla_j\left(R^2\right)_{ik}\right)\right]\\
&\quad -\frac{1}{2}g_{ij}\left[R^k_lR^l_mR^m_k-3\nabla_k\nabla_l\left(R^2\right)^{kl}\right],\\
(F_7)_{ij} &= 2\nabla_i\nabla_j\left(\nabla^2R\right)-2\left(\nabla^2R\right)R_{ij}\\
&\quad +\left(\nabla_iR\right)\left(\nabla_jR\right)-\frac{1}{2}g_{ij}\left[\left(\nabla R\right)^2+4\nabla^4R\right],\\
(F_8)_{ij} &= \nabla^4R_{ij}-\nabla_k\left(\nabla_i\nabla^2R^k_j+\nabla_j\nabla^2R^k_i\right)\\
&\quad -\left(\nabla_iR^k_l\right)\left(\nabla_jR^l_k\right)-2\left(\nabla^kR^l_i\right)\left(\nabla_kR_{jl}\right)\\
&\quad -\frac{1}{2}g_{ij}\left[\left(\nabla_kR_{lm}\right)^2-2\left(\nabla_k\nabla_l\nabla^2R^{kl}\right)\right], \qquad \text{(A.1)}\\
F^{ij}_{(\varphi,1)} &= \frac{1}{2}\varphi\left\{\left(2K+\nabla^2\varphi\right)R^{ij}-2\left(2K^j_k+\nabla^j\nabla_k\varphi\right)R^{ik}\right.\\
&\quad -2\left(2K^i_k+\nabla^i\nabla_k\varphi\right)R^{jk}\\
&\quad \left.-\left(2\Lambda_g-R\right)\left(2K^{ij}+\nabla^i\nabla^j\varphi\right)\right\},\\
F^{ij}_{(\varphi,2)} &= \frac{1}{2}\nabla_k\left\{\varphi\mathcal{G}^{ik}\left(\frac{2N^j}{N}+\nabla^j\varphi\right)\right.\\
&\quad \left.+\varphi\mathcal{G}^{jk}\left(\frac{2N^i}{N}+\nabla^i\varphi\right)-\varphi\mathcal{G}^{ij}\left(\frac{2N^k}{N}+\nabla^k\varphi\right)\right\},\\
F^{ij}_{(\varphi,3)} &= \frac{1}{2}\left\{2\nabla_k\nabla^{(i}f^{j)k}_\varphi-\nabla^2f^{ij}_\varphi-\left(\nabla_k\nabla_lf^{kl}_\varphi\right)g^{ij}\right\},
\end{aligned}\tag{A.2}$$

where

$$f^{ij}_\varphi = \varphi\left\{\left(2K^{ij}+\nabla^i\nabla^j\varphi\right)-\frac{1}{2}\left(2K+\nabla^2\varphi\right)g^{ij}\right\}.\tag{A.3}$$

Appendix B: Function $F^{ij}_{(\varphi,n)}$ in Spherical Static Spacetimes

In the spherically symmetric static spacetimes described by the metric (3.1), the function $F^{ij}_{(\varphi,n)}$ defined by Eq.(A.2) is given by

$$\begin{aligned}
\left[F_{(\varphi,1)}\right]_{ij} &= \frac{\varphi e^{-2\nu}}{r^2}\left\{\left[e^{2\nu}\left(1-\Lambda_gr^2\right)-\left(1+r\nu'\right)\right]\varphi''\right.\\
&\quad +\left[e^{2\nu}\left(\Lambda_gr^2-1\right)+\left(3+r\nu'\right)\right]\nu'\varphi'\\
&\quad -2e^{\mu+\nu}\left[e^{2\nu}\left(\Lambda_gr^2-1\right)+1\right]\mu'\\
&\quad \left.+2e^{\mu+\nu}\left(2-r\mu'\right)\nu'\right\}\delta^r_i\delta^r_j\\
&\quad +\frac{1}{2}\varphi e^{-4\nu}\left\{\left[r\nu'-\left(1-e^{2\nu}\right)\right]\varphi''\right.\\
&\quad +\left(3-r\nu'-e^{2\nu}\right)\nu'\varphi'-2\Lambda_gre^{2\nu}\varphi'\\
&\quad -2e^{\mu+\nu}\left(2-r\nu'-e^{2\nu}\right)\mu'\\
&\quad \left.+4e^{\mu+\nu}\left(\nu'-\Lambda_gre^{2\nu}\right)\right\}\Omega_{ij},\\
\left[F_{(\varphi,2)}\right]_{ij} &= \frac{e^{-2\nu}}{2r^2}\left\{\left[e^{2\nu}\left(\Lambda_gr^2-1\right)+1\right]\left[\varphi\varphi''\right.\right.\\
&\quad \left.+\left(\varphi'+\varphi\nu'\right)\varphi'+2e^{\mu+\nu}\left(\varphi'+\varphi\mu'\right)\right]\\
&\quad +4\varphi e^{\mu+\nu}\left(\nu'-\Lambda_gre^{2\nu}\right)\\
&\quad \left.-2\Lambda_gr\varphi e^{2\nu}\varphi'\right\}\delta^r_i\delta^r_j\\
&\quad +\frac{1}{2}e^{-4\nu}\left\{r\varphi\left(\nu'-\Lambda_gre^{2\nu}\right)\varphi''\right.\\
&\quad +r\varphi\left(\varphi'+2e^{\mu+\nu}\right)\nu''\\
&\quad +r\left(\nu'-\Lambda_gre^{2\nu}\right)\varphi'^2\\
&\quad -r\varphi\left(3\varphi'+4e^{\mu+\nu}\right)\nu'^2
\end{aligned}$$

$$
\begin{aligned}
&- \left(\varphi - 2re^{\mu+\nu} - \Lambda_g r^2 \varphi e^{2\nu}\right)\nu'\varphi' \\
&- 2e^{\mu+\nu}\left(\Lambda_g r^2 e^{2\nu}\varphi' + \varphi\nu'\right) \\
&\left. + 2r\varphi e^{\mu+\nu}\left(\nu' - \Lambda_g r e^{2\nu}\right)\mu' \right\}\Omega_{ij},
\end{aligned}
$$

$$
\begin{aligned}
\left[F_{(\varphi,3)}\right]_{ij} &= \frac{e^{-2\nu}}{r^2}\Bigg[r\varphi\nu'\varphi'' + \varphi'^2 - \varphi\left(r\nu'+2\right)\nu'\varphi' \\
&\quad + 2e^{\mu+\nu}\left(\varphi' + r\varphi\nu'\mu' - 2\varphi\nu'\right)\Bigg]\delta^r_i\delta^r_j \\
&+ \frac{1}{2}e^{-4\nu}\Bigg\{2r\left(\varphi' - \varphi\nu' + e^{\mu+\nu}\right)\varphi'' \\
&\quad - r\varphi\left(\varphi' + 2e^{\mu+\nu}\right)\nu'' \\
&\quad + \left[4r\left(\varphi\nu' - \varphi'\right) + \varphi\right]\nu'\varphi' \\
&\quad + e^{\mu+\nu}\Big[4r\varphi\left(\nu' - \mu'\right)\nu' + 2\varphi\nu' \\
&\qquad + 2r\left(\mu' - 3\nu'\right)\varphi'\Big]\Bigg\}\Omega_{ij}. \qquad \text{(B.1)}
\end{aligned}
$$

Appendix C: Impulsive Parts of $F^{ij}_{(\varphi,n)}$

From Eqs.(7.1) and (7.4) we find that $\left[F_{(\varphi,n)}\right]_{ij}$ given by Eq.(B.1) takes the form,

$$
\left[F_{(\varphi,n)}\right]_{ij} = \left[F_{(\varphi,n)}\right]^D_{ij} + \left[F^{Im}_{(\varphi,n)}\right]_{ij}\delta(r-r_0), \qquad \text{(C.1)}
$$

where

$$
\begin{aligned}
\left[F^{Im}_{(\varphi,1)}\right]_{ij} &= \frac{\varphi e^{-2\nu}}{r^2}\left[e^{2\nu}\left(1-\Lambda_g r^2\right) - \left(1 + r\nu'\right)\right] \\
&\quad \times \left[\varphi'\right]^- \delta^r_i\delta^r_j \\
&\quad + \frac{1}{2}\varphi e^{-4\nu}\left[r\nu' - \left(1 - e^{2\nu}\right)\right]\left[\varphi'\right]^-\Omega_{ij}, \\
\left[F^{Im}_{(\varphi,2)}\right]_{ij} &= \frac{\varphi e^{-2\nu}}{2r^2}\left[e^{2\nu}\left(\Lambda_g r^2 - 1\right) + 1\right]\left[\varphi'\right]^- \delta^r_i\delta^r_j \\
&\quad + \frac{1}{2}r\varphi e^{-4\nu}\Bigg[\left(\nu' - \Lambda_g r e^{2\nu}\right)\left[\varphi'\right]^- \\
&\qquad + \left(\varphi' + 2e^{\mu+\nu}\right)\left[\nu'\right]^-\Bigg]\Omega_{ij}, \\
\left[F^{Im}_{(\varphi,3)}\right]_{ij} &= \frac{\varphi\nu' e^{-2\nu}}{r}\left[\varphi'\right]^- \delta^r_i\delta^r_j \\
&\quad + \frac{1}{2}re^{-4\nu}\Bigg[2\left(\varphi' - \varphi\nu' + e^{\mu+\nu}\right)\left[\varphi'\right]^- \\
&\qquad - \varphi\left(\varphi' + 2e^{\mu+\nu}\right)\left[\nu'\right]^-\Bigg]\Omega_{ij}. \qquad \text{(C.2)}
\end{aligned}
$$

[1] P. Horava, Phys. Rev. D**79**, 084008 (2009) [arXiv:0901.3775].

[2] E.M. Lifshitz, Zh. Eksp. Toer. Fiz. **11**, 255; 269 (1941).

[3] Y.-Q. Huang, A. Wang, and Q. Wu, Mod. Phys. Lett. **25**, 2267 (2010) [arXiv:1003.2003].

[4] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973), pp.484-528.

[5] S. Mukohyama, arXiv:1007.5199.

[6] T. Sotiriou, M. Visser, and S. Weinfurtner, Phys. Rev. Lett. **102**, 251601 (2009) [arXiv:0904.4464]; J. High Energy Phys., **10**, 033 (2009) [arXiv:0905.2798]; arXiv:1002.0308.

[7] A. Wang and R. Maartens, Phys. Rev. D **81**, 024009 (2010) [arXiv:0907.1748]; A. Wang, D. Wands, and R. Maartens, J. Cosmol. Astropart. Phys., **03**, 013 (2010) [arXiv:0909.5167].

[8] A. Wang and Q. Wu, arXiv:1009.0268.

[9] T.P. Sotiriou, arXiv:1010.3218.

[10] P. Horava and C.M. Melby-Thompson, Phys. Rev. D**82**, 064027 (2010) [arXiv:1007.2410].

[11] C. Teitelboim, *The Hamiltonian Structure of Space-time*, in *General Relativity and Gravitation*, Vol. 1, edited by A. Held (Plenum Press, 1980); M. Henneaux, *Zero Hamiltonian Signature Spacetimes*, Bull. Soc. Math. Belg. **31**, 47 (1979).

[12] A.M. da Silva, arXiv:1009.4885.

[13] R.A. Konoplya, arXiv:0905.1523; T. Harko, Z. Kovacs, and F.S.N. Lobo, arXiv:0908.2874; L. Iorio and M.L. Ruggiero, arXiv:0909.2562; arXiv:0909.5355.

[14] K. Izumi and S. Mukohyama,Phys. Rev. D**81**, 044008 (2010) [arXiv:0911.1814].

[15] A. Wang and Y. Wu, arXiv:1009.2089 (Paper I).

[16] J. Greenwald, A. Papazoglou, and A. Wang, Phys. Rev. D**81**, 084046 (2010) [arXiv:0912.0011] (Paper II).

[17] N.O. Santos, Mon. Not. R. Astron. Soc. **216**, 403 (1985); W.B. Bonnor, A.K. G. de Oliveira, and N.O. Santos, Phys. Rept. **181**, 269 (1989).

[18] P. Anninos, "Computational Cosmology: From The Early Universe To The Large Scale Structure," Living Rev. Rel. **1**, 9 (1998).

[19] P. Painleve, C. R. Acad. Sci. (Paris) **173**, 677 (1921); A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**, 1 (1922).

[20] A.S. Eddington, The Mathematical Theory of Relativity (Cambridge University Press, Cambridge, 1957).

[21] J.B. Hartle, Gravity, An Introduction to Einstein's General Relativity (Addison Wesley, New York, 2003).

[22] B. Bertotti, L. Iess and P. Tortora, Nature, **425**, 374 (2003).

[23] J.G. Williams, S.G. Turyshev, and D.H. Boggs, Phys.

Rev. Lett. **93**, 261101 (2004).
[24] S. Reynaud and M.-T. Jaekel, arXiv:0801.3407.
[25] J. Alexandre and P. Pasipoularides, arXiv:1010.3634.
[26] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[27] D. Blas, O.Pujolas, and S. Sibiryakov, Phys. Lett. B**688**, 350 (2010) [arXiv:0912.0550]; arXiv.1007.3503.
[28] C. Charmousis, G. Niz, A. Padilla, and P.M. Saffin, J. High Energy Phys., **08**, 070 (2009) [arXiv:0905.2579]; D. Blas, O. Pujolas, and S. Sibiryakov, J. High Energy Phys., **03**, 061 (2009) [arXiv:0906.3046]; K. Koyama and F. Arroja, J. High Energy Phys., **03**, 061 (2010) [arXiv:0910.1998]; A. Papazoglou and T.P. Sotiriou, Phys. Lett. B**685**, 197 (2010) [arXiv:0911.1299]; I. Kimpton and A. Padilla, J. High Energy Phys. **07**, 014 (2010) [arXiv:1003.5666]; A. Padilla, arXiv:1009.4074.
[29] We thank S. Mukohyama for pointing out this.